**SOLUTIONS FOR ADMISSIONS TEST IN**
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
**WEDNESDAY 2 NOVEMBER 2016**

**Mark Scheme:**

Each part of Question 1 is worth 4 marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

**QUESTION 1:**

**A.** Considering the sequence, $a_2 = l$, $a_3 = l^2$, $a_4 = l^3$, each additional term multiplies the previous term by $l$. The product of the first 15 terms is equal to $l^{1+2+\ldots+14} = l^{\frac{14*15}{2}} = l^{105}$. **The answer is (d).**

**B.** Call the length of one of the sides of the hexagon $p$, then the side of the square is equal to $p + (1 - p) = 1$. Then as the hexagon side forms a triangle in each corner of the square, using Pythagoras, $p^2 = (1 - p)^2 + (1 - p)^2$. Solving this quadratic results in $p = 2 \pm \sqrt{2}$, but as the length must be less than 1 **the answer is (b).**

**C.** We can rewrite the given equation as $(x + \frac{a}{2})^2 + (y + \frac{b}{2})^2 = c + \frac{a^2}{4} + \frac{b^2}{4}$. For the circle to contain the origin, the distance from the centre to the origin must be less than the radius, so $\frac{a^2}{4} + \frac{b^2}{4} < c + \frac{a^2}{4} + \frac{b^2}{4}$. **The answer is (a).**

**D.** $cos^n(x) + cos^{2n}(x) = cos^n(x)(1 + cos^n(x)) = 0$. For this to be true, if $n$ is even, $cos(x) = 0$ has two roots, but when $n$ is odd either $cos(x) = 0$ or $cos(x) = -1$, which is three roots. Hence **the answer is (d).**

**E.** When $x = 0$, $y = 1 - 1 = 0$, so we can rule out (d) and (e). To work out the number of $x$-axis intersection points, consider $(x - 1)^2 = cos(\pi x)$. The shape of these graphs means they cannot intersect 6 times (eliminating (b)). The answer cannot be (c), because we know there is a crossing point $x = 2$, but that $y$ is positive when $x = 1$. So **the answer is (a).**

**F.** Using the factor theorem, for $(x^2 + 1)$ to be a factor, $(x^2 + 1) = 0$, so $x^2 = -1$. Then the equation given becomes $(4)^n - (2)^n(-2)^n = 0$. This only holds when $(-2)^n$ is positive, so **the answer is (b)**.

**G.** Considering the first few terms $x_0 = 1$, $x_1 = x_0 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 8$, and so on. By observation, $x_n = 2^{n-1}$ for $n \geqslant 1$. As this is a geometric progression, we can evaluate the sum of the sequence as

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{x_k} &= \frac{1}{1} + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \\ &= 1 + \frac{1}{1 - \frac{1}{2}} \\ &= 3\end{aligned}$$

**The answer is (d).**

**H.** The area bounded by the $x$-axis and the curve $y = f(x)$, $A_1$ is equal to

$$A_1 = \int_{-\sqrt{a}}^{\sqrt{a}} f(x)\,\mathrm{d}x = \frac{4}{3}a^{\frac{3}{2}},$$

whilst the area bounded by the $x$-axis and the curve $y = g(x)$, $A_2$ is equal to

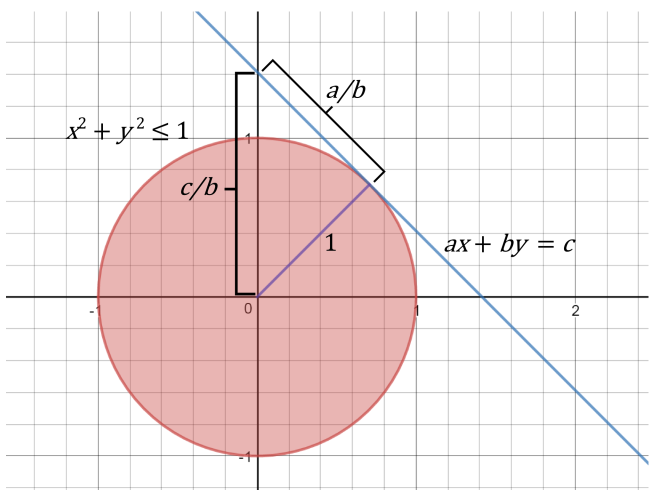
$$A_2 = |\int_{-\sqrt[4]{a}}^{\sqrt[4]{a}} g(x)\,\mathrm{d}x| = \frac{8}{5}a^{\frac{5}{4}}.$$

We require an $a$ such that $A_1 > A_2$, so

$$\begin{aligned} \frac{4}{3}a^{\frac{3}{2}} &> \frac{8}{5}a^{\frac{5}{4}} \\ 20a^{\frac{6}{4}} &> 24a^{\frac{5}{4}} \\ a^{\frac{1}{4}} &> \frac{6}{5} \end{aligned}$$

and so **the answer is (e)**.

**I.** Let $ax + by = c$, which rearranges to $y = \frac{a}{b}x + \frac{c}{b}$. Given that $b$ is positive we can interpret this as achieving the maximum $c$ when the line $y = \frac{a}{b}x + \frac{c}{b}$ is moved up the $y$-axis whilst still intersecting the disc formed by $x^2 + y^2 \leqslant 1$. Hence the line should be tangent to the unit circle.




By Pythagoras,

$$\left(\frac{c}{b}\right)^2 = 1 + \left(\frac{a}{b}\right)^2$$

and so **the answer is (c).**

**J.** We are trying to construct counter-examples for each of the statements. Note that $0 \leqslant x(n) \leqslant 9$. (a) is true since, for example, $\Pi(4) = 1$, but 4 is not prime. (b) is false - we don't need to consider even $n$ beyond $x_n = 4$; for this case we know no primes end in a 4, but for example $\Pi(64) = 1$ as $64 = 2^6$. For odd $n$, $x(n) = 1, \Pi(n) = 1$, counterexample $n = 121 = 11^2$; $x(n) = 3, \Pi(n) = 1$, counterexample $n = 243 = 3^5$; $x(n) = 5, \Pi(n) = 1$, counterexample $n = 25 = 5^2$; $x(n) = 7, \Pi(n) = 1$, counterexample $n = 16807 = 7^5$; $x(n) = 9, \Pi(n) = 1$, counterexample $n = 9 = 3^2$. (c), (d), and (e) are all true. **The answer is (b).**

**2.** (i) [1 mark] We have

$$\begin{aligned} A(B(x)) &= 2(3x+2)+1 = 6x+5; \\ B(A(x)) &= 3(2x+1)+2 = 6x+5. \end{aligned}$$

(ii) [3 marks] We note

$$\begin{aligned} A^2(x) &= 2(2x+1)+1 = 4x+2+1, \\ A^3(x) &= 2(4x+2+1)+1 = 8x+4+2+1, \end{aligned}$$

and so more generally

$$A^n(x) = 2^n x + 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = 2^n x + (2^n - 1)$$

using the geometric series formula (pattern spotting sufficient).

(iii) [4 marks] As $108 = 2^2 3^3$ then $F$ can be achieved using two applications of $A$ and three applications of $B$. As $AB = BA$ then only one such $F$ can be achieved but the number of different orders in which $A, A, B, B, B$ might be performed is ${}^5C_2 = 10$.

(iv) [3 marks] Note that in each case the constant coefficient is one less than the coefficient of $x$. We can prove this by noting

$$\begin{aligned} A(ax+(a-1)) &= 2(ax+(a-1))+1 = 2ax+2a-1; \\ B(ax+(a-1)) &= 3(ax+(a-1))+2 = 3ax+3a-1. \end{aligned}$$

So $c = 107$. [Alternatively to find $c$ a student might just determine $A^2B^3$.]

[Alternative: Commuting argument:
By part (i) $A$ and $B$ commute. Therefore we only need to check 1 of the possible configurations. From this calculation we find that $c = 107$.]

(v) [4 marks] As each $A^{m_i}B^{n_i}(x)$ will have a constant coefficient one less than its $x$ coefficient it follows that $k = 214 - 92 = 122$. However the $x$ coefficient of $A^{m_i}B^{n_i}(x)$ can never be less than 2 so the sum of 122 such functions cannot have an $x$ coefficient less than 244.

[Alternative: Divisible by 6 argument:
Each term contains at least an $A$ and at least a $B$, and so each $x$ coefficient is a multiple of 6. However 214 is not divisible by 6 and hence there exist no positive integers.]

**3. 3. Solution:**
(i) [1 mark] Note that

$$f(2\alpha - x) = (2\alpha - x - \alpha)^2 = (\alpha - x)^2 = (x - \alpha)^2 = f(x)$$

for all $x$ and hence $f$ is bilateral.

(ii) [2 marks] Consider, for example, $x = \alpha + 1$ where

$$f(\alpha + 1) = 1 \neq -1 = f(\alpha - 1) = f(2\alpha - (\alpha + 1)).$$

It follows that $f$ is not bilateral.

(iii) [2 marks] Note that

$$\begin{aligned}\int_a^b x^n \,\mathrm{d}x = \left[\frac{x^{n+1}}{n+1}\right]_a^b &= \frac{b^{n+1} - a^{n+1}}{n+1} \\ &= -\left(\frac{a^{n+1} - b^{n+1}}{n+1}\right) = -\left[\frac{x^{n+1}}{n+1}\right]_b^a = -\int_b^a x^n \,\mathrm{d}x\end{aligned}$$

as required.

[Alternatively: Some students may show this graphically and argue that area is preserved under reflection]

(iv) [3 marks] Since $f$ is a polynomial there is a non-negative integer $d$ and reals $c_0, \ldots, c_d$ such that $f(x) = c_0 + c_1 x + \cdots + c_d x^d$ for all $x$. Integration is linear so by the previous part we have

$$\begin{aligned}\int_a^b f(x) \,\mathrm{d}x &= \sum_{i=0}^{d} c_i \int_a^b x^i \,\mathrm{d}x \\ &= -\sum_{i=0}^{d} c_i \int_b^a x^i \,\mathrm{d}x = -\int_b^a f(x) \,\mathrm{d}x\end{aligned}$$

as required.

(v) [2 marks] The first integral is just the signed area under the graph of $y = f(x)$ between $\alpha$ and $t$ and the second integral is the signed area under the graph of $y = f(x)$ between $2\alpha - t$ and $\alpha$. The second signed area is a reflection of the first, and area is preserved under reflection. Hence the integrals are equal.

(vi) [3 marks] For $t \geq \alpha$ we have by the previous two parts that

$$G(t) = \int_\alpha^t f(x) \,\mathrm{d}x = \int_{2\alpha - t}^{\alpha} f(x) \,\mathrm{d}x = -\int_\alpha^{2\alpha - t} f(x) \,\mathrm{d}x = -G(2\alpha - t).$$

If $t \leq \alpha$ then put $u = 2\alpha - t \geq \alpha$ and note that by what we have just shown

$$G(2\alpha - t) = G(u) = -G(2\alpha - u) = -G(t).$$

The result follows.

(vii) [2 marks] Since $f$ is a bilateral polynomial we see $G(2\alpha - t) = -G(t)$ for all $t$. On the other hand since $G$ is bilateral we have $G(2\alpha - t) = G(t)$ for all $t$, so $G(t) = 0$ for all $t$ as required.

**4.** (i) [3 marks] Let $d_1$ be the distance from $(0,0)$ to the point where $C_1$ touches the $x$-axis. Note that the $x$-axis is tangent to $C_1$ and hence perpendicular to the radius at this point. So $C_1$ has centre $(d_1, 1)$. We have a right-angled triangle, with $\frac{1}{d_1} = \tan(\alpha)$, so $d_1 = \frac{1}{\tan(\alpha)}$.
So the centre of $C_1$ is $(\frac{1}{\tan(\alpha)}, 1)$.

(ii) [1 mark] $C_1$ has centre $(\frac{1}{\tan(\alpha)}, 1)$ and radius 1, so has equation

$$(x - \frac{1}{\tan(\alpha)})^2 + (y-1)^2 = 1.$$

(iii) [3 marks] Let $d_2$ be the distance between the points where $C_1$ and $C_2$ touch the $x$-axis. Then Pythagoras on the right-angled triangle gives $(1+3)^2 = 2^2 + d_2^2$, so $d_2^2 = 12$.
Also we have similar triangles (both have a right angle and share angle $\alpha$) so

$$\frac{3}{1} = \frac{d_2 + d_1}{d_1}$$

so $d_2 = 2d_1$.
So $12d_2^2 = (2d_1)^2 = 4d_1^2$, so $d_1 = \sqrt{3}$ (must have $d_1 > 0$). So $\tan(\alpha) = \frac{1}{d_1} = \frac{1}{\sqrt{3}}$ so $\alpha = 30°$ (or $\frac{\pi}{6}$).

(iv) [3 marks] Take $\alpha = 30°$. Let $C_3$ have radius $r$. Let $d_3$ be the distance between the points where $C_2$ and $C_3$ touch the $x$-axis. Then by similar triangles we have

$$\frac{r}{d_1 + d_2 + d_3} = \frac{1}{d_1} = \frac{1}{\sqrt{3}}.$$

So $r = \frac{d_1+d_2+d_3}{\sqrt{3}} = 3 + \frac{d_3}{\sqrt{3}}$. So $d_3 = \sqrt{3}(r-3)$.
Also since $C_2$ and $C_3$ touch Pythagoras gives

$$(r+3)^2 = (r-3)^2 + d_3^2 = (r-3)^2 + 3(r-3)^2 = 4(r-3)^2,$$

so $r^2 + 6r + 9 = 4r^2 - 24r + 36$, which factorises to $(r-1)(r-9) = 0$.
We're looking for $C_3$ larger than $C_2$ so $r = 9$.

(v) [5 marks] Centres of triangle $C_1$ and $C_2$ are $(\frac{1}{\tan(\alpha)}, 1)$ and $(\frac{3}{\tan(\alpha)}, 3)$ respectively. Area of trapezium is half (bottom plus top) times height, so:

$$\frac{3+1}{2}\frac{2}{\tan(\alpha)} = \frac{4}{\tan(\alpha)}.$$

Or break down as rectangle (area $= \frac{2}{\tan(\alpha)}$) plus triangle (area $= \frac{2}{\tan(\alpha)}$).
Now deduct $C_1$ sector and $C_2$ sector from trapezium area. Area of $C_1$ sector is $\frac{1}{2}1^2(\frac{\pi}{2} + \alpha) = \frac{\pi}{4} + \frac{\alpha}{2}$.
Area of $C_2$ sector is $\frac{1}{2}3^2(\frac{\pi}{2} - \alpha) = \frac{9\pi}{4} - \frac{9\alpha}{2}$.

So interstitial area is:

$$\frac{4}{\tan(\alpha)} - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \left(\frac{9\pi}{4} - \frac{9\alpha}{2}\right) = \frac{4}{\tan(\alpha)} - \frac{5\pi}{2} + 4\alpha = 4\sqrt{3} - \frac{11\pi}{6}$$

**5.** (i) [3 marks] We have

$$\begin{aligned} s_1 &= 2(A+B)+C = 2 \\ s_2 &= 4(2A+B)+C = 10 \\ s_3 &= 8(3A+B)+C = 34 \end{aligned}$$

(ii) [3 marks] Subtracting the first equation from the other two gives

$$\begin{aligned} 6A+2B &= 8 \\ 22A+6B &= 32, \end{aligned}$$

whence $4A = 8$, so $A = 2$, $B = -2$, $C = 2$ and $f(n) = (n-1)2^{n+1} + 2$.

(iii) [2 marks] We have

$$\begin{aligned} s_{k+1} &= f(k) + (k+1)2^{k+1} \\ &= (k-1)2^{k+1} + 2 + (k+1)2^{k+1} \\ &= k2^{k+2} + 2 = f(k+1) \end{aligned}$$

as required.

(iv) [4 marks] We have

$$\begin{aligned} t_n &= (n + 2n + 4n + \cdots + 2^n.n) - (2 + 8 + 24 + ... + 2^n.n) \\ &= n(2^{n+1} - 1) - f(n) \\ &= n(2^{n+1} - 1) - (n-1)2^{n+1} - 2 \\ &= 2^{n+1} - n - 2. \end{aligned}$$

Now $u_n = t_n/2^n$, so

$$u_n = 2 - \frac{n+2}{2^n}.$$

(v) [3 marks]

$$\begin{aligned} \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} (2k2^k - 2^{k+1} + 2) \\ &= \sum_{k=1}^{n} (k2^{k+1}) - \sum_{k=1}^{n} (2^{k+1}) + 2n \\ &= 2\sum_{k=1}^{n} (k2^k) - 2^{n+2} + 4 + 2n \\ &= 2f(n) - 2^{n+2} + 4 + 2n \\ &= 2^{n+2}n - 2^{n+3} + 2n + 8 \end{aligned}$$

**6.** (i) [3 marks] There are no possible arrangements - if $A$ is a 1, then either $B$ and $D$ are both 1s or both 0s. However, if $B$ and $D$ are both 1s then $C$ must also be a 1 - but that would require all the dancers to be 1s which is forbidden. If $B$ and $D$ are both 0s then $C$ must also be a 0 otherwise $D$ would not be off-beat. But if $C$ is a 0 they cannot be off-beat.

(ii) [3 marks] Assume that $A$ is a 1 and holds hands with $F$ and $B$, then either $F$ and $B$ are both 1s or both 0s. If both $F$ and $B$ are 1s then this pattern must propagate around the circle, forcing everyone to be 1s, which is forbidden. If $F$ and $B$ are both 0s then $C$ and $E$ must also be 0s, to keep $F$ and $B$ off-beat. However to ensure $C$, $D$, and $E$ are off-beat $D$ must be a 1. Hence the only possible arrangements are those where precisely two dancers on opposite positions on the ring are 1, and there are 3 such arrangements.

(iii) [3 marks] Each person holding hands either requires one of the three dancers to be a 1 or all three to be a 1. If all three, then this propagates round resulting in all 1s, which is forbidden. Thus for each triplet of dancers one person is a 1. Then either spot the 1,0,0 pattern which only repeats when $n$ is a multiple of three, or look at the sum of each local triplet of dancers which must be equal to $n$ and also equal to $3k$ where $k$ is the number of dancers who are 1s.

(iv) [2 marks] If $n$ is even two separate rings form, however each ring can only be off-beat if the number of dancers are a multiple of 3, by previous argument. If $n$ is odd, then $n$ must be a multiple of 3 still because a ring is still formed (with displaced dancers).

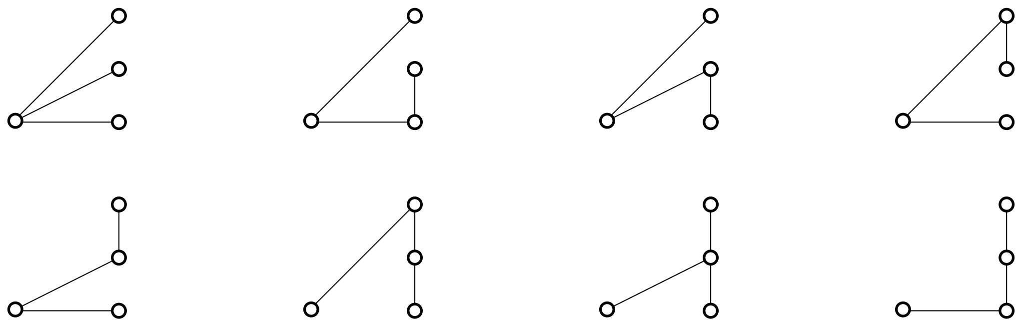
(v) [1 mark] Either one dancer is a 1 or three dancers are 1s and one is a 0. There are 8 different ways in total.

(vi) [3 marks] There must be at least one dancer who is a 1. Holding hands with this dancer there must be either no dancers or precisely two dancers who are 1s. If none of the dancers are 1s, then the alternating 0, 1 pattern is very obvious. If two dancers are 1s, then this leads to a situation where all dancers are 1s, which is still forbidden. Hence there are 2 possible ways of arranging off-beat dances.

**7.** (Example taken from Graham, Knuth, Patashnik, *Concrete Mathematics*.)

(i) [2 marks] Three 2-spans:

(ii) [4 marks] Eight 3-spans:



(iii) [3 marks] In a 4-span, the top group may have $t = 1$, 2, 3 or 4 elements, and may be connected to the hub by any of $t$ line segments in each case. If $t = 4$, that is the end of the story, but if $t < 4$ then the remaining tips may form any $(4 - t)$-span. Thus (using the notation of the next part),

$$z_4 = 1.z_3 + 2.z_2 + 3.z_1 + 4 = 1 \times 8 + 2 \times 3 + 3 \times 1 + 4 = 21.$$

(iv) [4 marks] More generally, we have

$$z_n = 1.z_{n-1} + 2.z_{n-2} + \cdots + (n-1).z_1 + n.$$

It follows that

$$z_5 = 1 \times 21 + 2 \times 8 + 3 \times 3 + 4 \times 1 + 5 = 55.$$

(v) [2 marks]

$$z_6 = 1 \times 55 + 2 \times 21 + 3 \times 8 + 4 \times 3 + 5 \times 1 + 6 = 144.$$